Motion of an Artificial Earth Satellite in an Orbit of Small Eccentricity

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The subject of this paper is the calculation of perturbations of the elements of near-circular earth satellite orbits Literal expressions are found for calculating first-order perturbations to an accuracy of the order of and including the first power of eccentricity The improvement of the elements of near-circular orbits is also discussed One section deals with the determination of constants of integration

1 The Differential Equations of Motion

LET a satellite of negligible mass be moving in the earth's gravitational field Lagrange's equations for determining the osculating elements have the form[3]

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial \epsilon}$$

$$\frac{de}{dt} = -\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \pi} - \frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\frac{1}{na^2}\frac{\partial R}{\partial \epsilon}$$

$$\frac{di}{dt} = \frac{-\operatorname{cosec} i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \Omega} - \frac{\tan(i/2)}{na^2\sqrt{1-e^2}}\left(\frac{\partial R}{\partial \pi} + \frac{\partial R}{\partial \epsilon}\right) \quad (1)$$

$$\frac{d\Omega}{dt} = \frac{\operatorname{cosec} i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i}$$

$$\frac{d\pi}{dt} = \frac{\tan(i/2)}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e}$$

$$\frac{d\epsilon}{dt} = -\frac{2}{na}\frac{\partial R}{\partial a} + \frac{\tan(i/2)}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} + \frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\frac{1}{na^2}\frac{\partial R}{\partial e}$$

where $a,e,i,\Omega,\pi = \omega + \Omega$, and ϵ are six of the orbital elements of the satellite; n is the mean motion; and R is the disturbing function In the case of orbits of small eccentricity these equations have certain drawbacks, since the small divisor e appears on the right-hand sides of the expressions for de/dt and $d\pi/dt$ To avoid this difficulty we transform Eqs (1) We replace e,π, and ϵ by the new elements

$$\lambda = \epsilon + \int_{t_0}^{t} n dt - \Omega$$

$$h = e\sin(\pi - \Omega)$$

$$l = e\cos(\pi - \Omega)$$

whence

$$\lambda_0 = \epsilon - \Omega$$

We first find

$$\frac{dh}{dt} = \sin\omega\frac{de}{dt} + e\cos\omega\frac{d\omega}{dt}$$

$$\frac{dl}{dt} = \cos\omega\frac{de}{dt} - e\sin\omega\frac{d\omega}{dt}$$

$$\frac{\partial R}{\partial e} = \sin\omega\frac{\partial R}{\partial h} + \cos\omega\frac{\partial R}{\partial l}$$

$$\frac{\partial R}{\partial \pi} = e\cos\omega\frac{\partial R}{\partial h} - e\sin\omega\frac{\partial R}{\partial l}$$

and

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\left[\frac{\partial R}{\partial h}\sin\omega + \frac{\partial R}{\partial l}\cos\omega\right] - \frac{\operatorname{cotan} i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i}$$

Using (1) we obtain

$$\frac{dh}{dt} = \frac{\sqrt{1-h^2-l^2}}{na^2}\frac{\partial R}{\partial l} - \frac{\sqrt{1-h^2-l^2}}{na^2}\times \frac{h}{1+\sqrt{1-h^2-l^2}}\frac{\partial R}{\partial \epsilon} - \frac{l\operatorname{cotan} i}{na^2\sqrt{1-h^2-l^2}}\frac{\partial R}{\partial i}$$

$$\frac{dl}{dt} = -\frac{\sqrt{1-h^2-l^2}}{na^2}\frac{\partial R}{\partial h} - \frac{\sqrt{1-h^2-l^2}}{na^2}\times \frac{l}{1+\sqrt{1-h^2-l^2}}\frac{\partial R}{\partial \epsilon} + \frac{h\operatorname{cotan} i}{na^2\sqrt{1-h^2-l^2}}\frac{\partial R}{\partial i} \quad (2)$$

Treating h and l as small quantities, we transform Eq (2) by neglecting second-order terms,

$$\frac{dh}{dt} = \frac{1}{na^2}\frac{\partial R}{\partial l} - \frac{h}{2na^2}\frac{\partial R}{\partial \epsilon} - \frac{l\operatorname{cotan} i}{na^2}\frac{\partial R}{\partial i}$$

$$\frac{dl}{dt} = -\frac{1}{na^2}\frac{\partial R}{\partial h} - \frac{l}{2na^2}\frac{\partial R}{\partial \epsilon} + \frac{h\operatorname{cotan} i}{na^2}\frac{\partial R}{\partial i} \quad (3)$$

We then find

$$\frac{\partial R}{\partial \epsilon} = \frac{\partial R}{\partial \lambda_0}$$

$$\frac{\partial R}{\partial \Omega} = \frac{\partial R}{\partial h}\frac{dh}{d\Omega} + \frac{\partial R}{\partial l}\frac{dl}{d\Omega} + \frac{\partial R}{\partial \lambda_0}\frac{d\lambda_0}{d\Omega} + \left(\frac{\partial R}{\partial \Omega}\right) = -l\frac{\partial R}{\partial h} + h\frac{\partial R}{\partial l} - \frac{\partial R}{\partial \lambda_0} + \left(\frac{\partial R}{\partial \Omega}\right)$$

$$\frac{\partial R}{\partial \pi} + \frac{\partial R}{\partial \epsilon} = l\frac{\partial R}{\partial h} - h\frac{\partial R}{\partial l} + \frac{\partial R}{\partial \lambda_0}$$

Hence

$$\frac{di}{dt} = \frac{\operatorname{cotan} i}{na^2}\left(l\frac{\partial R}{\partial h} - h\frac{\partial R}{\partial l} + \frac{\partial R}{\partial \lambda_0}\right) - \frac{\operatorname{cosec} i}{na^2}\frac{\partial R}{\partial \Omega} \quad (4)$$

Finally,

$$\frac{d\lambda_0}{dt} = -\frac{2}{na}\frac{\partial R}{\partial a} + \frac{1}{2na^2}\left(h\frac{\partial R}{\partial h} + l\frac{\partial R}{\partial l}\right) - \frac{\operatorname{cotan} i}{na^2}\frac{\partial R}{\partial i} \quad (5)$$

In this way, Lagrange's equations as they appear in (1) can be replaced by the following equations:

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial \lambda_0}$$

$$\frac{dh}{dt} = \frac{1}{na^2}\frac{\partial R}{\partial l} - \frac{h}{2na^2}\frac{\partial R}{\partial \lambda_0} - \frac{l\operatorname{cotan} i}{na^2}\frac{\partial R}{\partial i} \quad (6)$$

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continuation of Eq (6)]

$$\frac{dl}{dt} = -\frac{l}{na^2}\frac{\partial R}{\partial h} - \frac{l}{2na^2}\frac{\partial R}{\partial \lambda_0} + \frac{h\ \mathrm{cotan}i}{na^2}\frac{\partial R}{\partial i}$$

$$\frac{d\Omega}{dt} = \frac{\mathrm{cosec}i}{na^2}\frac{\partial R}{\partial i}$$

$$\frac{di}{dt} = \frac{\mathrm{cotan}i}{na^2}\left(l\frac{\partial R}{\partial h} - h\frac{\partial R}{\partial l} + \frac{\partial R}{\partial \lambda_0}\right) - \frac{\mathrm{cosec}i}{na^2}\frac{\partial R}{\partial \Omega}$$

$$\frac{d\lambda_0}{dt} = -\frac{2}{na}\frac{\partial R}{\partial a} + \frac{1}{2na^2}\left(h\frac{\partial R}{\partial h} + l\frac{\partial R}{\partial l}\right) - \frac{\mathrm{cotan}i}{na^2}\frac{\partial R}{\partial i}$$

The equations obtained are valid for calculating perturbations of elements with any small eccentricity Specifically, for a circular orbit Eqs (6) take a very simple form:

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial \lambda_0}$$

$$\frac{dh}{dt} = \frac{1}{na^2}\frac{\partial R}{\partial l}$$

$$\frac{dl}{dt} = -\frac{1}{na^2}\frac{\partial R}{\partial h}$$

$$\frac{d\Omega}{dt} = \frac{\mathrm{cosec}i}{na^2}\frac{\partial R}{\partial i} \qquad (7)$$

$$\frac{di}{dt} = \frac{\mathrm{cotan}i}{na^2}\frac{\partial R}{\partial \lambda_0} - \frac{\mathrm{cosec}i}{na^2}\frac{\partial R}{\partial \Omega}$$

$$\frac{d\lambda_0}{dt} = -\frac{2}{na}\frac{\partial R}{\partial a} - \frac{\mathrm{cotan}i}{na^2}\frac{\partial R}{\partial i}$$

2 The Expansion of the Disturbing Function

The potential of an oblate spheroid at an external point has the form

$$R = -\tfrac{1}{3}J\ fm(a'^2/r^3)(3\sin^2\delta - 1) \qquad (8)$$

where f is the gravitational constant; m is the mass of the earth; r is the radius vector of the satellite; a' is the equatorial radius of the earth; δ is the inclination of the satellite; and J is a constant depending on the oblateness of the earth The expansion of R into a power series in terms of the satellite orbit eccentricity to the sixth power of the eccentricity was taken from the paper by Proskurin and Batrakov [2] Limiting ourselves to the first power of the eccentricity, we have

$$R_1 = \frac{1}{3}J\ fm\frac{a'^2}{a^3}\left(1 - \frac{3}{2}\sin^2 i\right)(1 + 3e\cos M) + \frac{1}{2}J\ fm\frac{a'^2}{a^3}\sin^2 i\left[-\frac{1}{2}e\cos(M + 2\omega) + \cos(2M + 2\omega) + \frac{7}{2}e\cos(3M + 2\omega)\right] \qquad (9)$$

In calculating the perturbations h and l it is necessary to include in the disturbing function terms of the second power in the eccentricity

$$R_2 = \frac{1}{3}J\ fm\frac{a'^2}{a^3}\left(1 - \frac{3}{2}\sin^2 i\right)\left(\frac{3}{2}e^2 + \frac{9}{2}e^2\cos 2M\right) + \frac{1}{2}J\ fm\frac{a'^2}{a^3}\sin^2 i\left[-\frac{5}{2}e^2\cos(2M + 2\omega) + \frac{17}{2}e^2\cos(4M + 2\omega)\right] \qquad (10)$$

Employing the new variables, we obtain

$$R_1 = J\ fm\frac{a'^2}{a^3}\left(1 - \frac{3}{2}\sin^2 i\right)\left(\frac{1}{3} + l\cos\lambda + h\sin\lambda\right) + J\ fm\frac{a'^2}{a^3}\sin^2 i\left(-\frac{1}{4}l\cos\lambda + \frac{1}{4}h\sin\lambda + \frac{1}{2}\cos 2\lambda + \frac{7}{4}l\cos 3\lambda + \frac{7}{4}h\sin 3\lambda\right) \qquad (11)$$

and

$$R_2 = \frac{1}{3}J\ fm\frac{a'^2}{a^3}\left(1 - \frac{3}{2}\sin^2 i\right)\left[\frac{3}{2}(h^2 + l^2) + \frac{9}{2}(l^2 - h^2)\cos 2\lambda + 9hl\sin 2\lambda\right] + \frac{1}{2}J\ fm\frac{a'^2}{a^3}\sin^2 i\left[-\frac{5}{2}(h^2 + l^2)\cos 2\lambda + \frac{17}{2}(l^2 - h^2)\cos 4\lambda + 17hl\sin 4\lambda\right] \qquad (12)$$

3 First-Order Perturbations

After computing the derivatives of the disturbing function with respect to each element and substituting the expressions obtained into Eqs (6), we find, after integration, the following formulas for calculating first-order perturbations of the orbital elements of a satellite to an accuracy limited by the first power of the eccentricity, or—which is the same thing—of h and l:

$$\delta a = 2J\left(\frac{a'^2}{a}\right)\left(1 - \frac{3}{2}\sin^2 i\right)(l\cos\lambda + h\sin\lambda) + J\left(\frac{a'^2}{a}\right)\sin^2 i\left(-\frac{1}{2}l\cos\lambda + \frac{1}{2}h\sin\lambda + \cos 2\lambda + \frac{7}{2}l\cos 3\lambda + \frac{7}{2}h\sin 3\lambda\right)$$

$$\delta h = J\left(\frac{a'}{a}\right)^2\left(1 - \frac{3}{2}\sin^2 i\right)\left(lnt + \sin\lambda + \frac{3}{2}l\sin 2\lambda - \frac{3}{2}h\cos 2\lambda\right) - \frac{1}{4}J\left(\frac{a'}{a}\right)^2\sin^2 i\left(\sin\lambda - \frac{7}{3}\sin 3\lambda + 5l\sin 2\lambda + h\cos 2\lambda - \frac{17}{2}l\sin 4\lambda + \frac{17}{2}h\cos 4\lambda\right) + J\left(\frac{a'}{a}\right)^2\cos^2 i\left(lnt - \frac{1}{2}l\sin 2\lambda\right) \qquad (13)$$

$$\delta l = J\left(\frac{a'}{a}\right)^2\left(1 - \frac{3}{2}\sin^2 i\right)\left(-hnt + \cos\lambda + \frac{3}{2}l\cos 2\lambda + \frac{3}{2}h\sin 2\lambda\right) - \frac{1}{4}J\left(\frac{a'}{a}\right)^2\sin^2 i\left(-\cos\lambda - \frac{7}{3}\cos 3\lambda - 5h\sin 2\lambda - \frac{17}{2}l\cos 4\lambda - \frac{17}{2}h\sin 4\lambda + l\cos 2\lambda\right) + J\left(\frac{a'}{a}\right)^2\cos^2 i\left(-hnt + \frac{h}{2}\sin 2\lambda\right)$$

$$\delta\Omega = -J\left(\frac{a'}{a}\right)^2\cos i\left(nt + \frac{7}{2}l\sin\lambda - \frac{5}{2}h\cos\lambda - \frac{1}{2}\sin 2\lambda - \frac{7}{6}l\sin 3\lambda - \frac{7}{6}h\cos 3\lambda\right)$$

$$\delta i = \frac{1}{2}J\left(\frac{a'}{a}\right)^2\sin i\cos i\left(-l\cos\lambda + h\sin\lambda + \cos 2\lambda + \frac{7}{3}l\cos 3\lambda + \frac{7}{3}h\sin 3\lambda\right)$$

$$\delta\lambda_0 = 2J\left(\frac{a'}{a}\right)^2\left(1 - \frac{3}{2}\sin^2 i\right)\left(nt + \frac{13}{4}l\sin\lambda -\right.$$

[continuation of Eq (13)]

$$\frac{13}{4} h \cos\lambda\Big) + 3J\left(\frac{a'}{a}\right)^2 \sin^2 i\Big(-\frac{13}{24} h \cos\lambda -$$
$$\frac{13}{24} l \sin\lambda - \frac{91}{72} h \cos 3\lambda + \frac{91}{72} l \sin 3\lambda + \frac{1}{2}\sin 2\lambda\Big) +$$
$$J\left(\frac{a'}{a}\right)^2 \cos^2 i\Big(nt + \frac{7}{2} l \sin\lambda - \frac{5}{2} h \cos\lambda -$$
$$\frac{1}{2}\sin 2\lambda - \frac{7}{6} l \sin 3\lambda + \frac{7}{6} h \cos 3\lambda\Big)$$

where on the right-hand sides we must insert the unperturbed elements a_0, h_0, $\lambda = \lambda_0 + n_0(t - t_0)$

For a circular orbit these formulas assume the form:

$$\delta a = J\,(a'^2/a)\sin^2 i \cos 2\lambda$$
$$\delta h = J\left(\frac{a'}{a}\right)^2\left(1 - \frac{3}{2}\sin^2 i\right)\sin\lambda - \frac{1}{4} J\left(\frac{a'}{a}\right)^2 \sin^2 i\left(\sin\lambda - \frac{7}{3}\sin 3\lambda\right)$$
$$\delta l = J\left(\frac{a'}{a}\right)^2\left(1 - \frac{3}{2}\sin^2 i\right)\cos\lambda + \frac{1}{4} J\left(\frac{a'}{a}\right)^2 \sin^2 i\left(\cos\lambda + \frac{7}{3}\cos 3\lambda\right) \quad (14)$$
$$\delta\Omega = -J\left(\frac{a'}{a}\right)^2 \cos i\left(nt - \frac{1}{2}\sin 2\lambda\right)$$
$$\delta i = \frac{1}{2} J\left(\frac{a'}{a}\right)^2 \sin i \cos i \cos 2\lambda$$
$$\delta\lambda_0 = 2J\left(\frac{a'}{a}\right)^2\left(1 - \frac{3}{2}\sin^2 i\right) nt + \frac{3}{2} J\left(\frac{a'}{a}\right)^2 \sin^2 i \sin 2\lambda + J\left(\frac{a'}{a}\right)^2 \cos^2 i\left(nt - \frac{1}{2}\sin 2\lambda\right)$$

4 Determining the Constants of Integration

The elements of the orbit of an artificial earth satellite are determined according to the formula

$$\begin{aligned} a &= a_0 + \delta a + c_1 \\ h &= h_0 + \delta h + c_2 \\ l &= l_0 + \delta l + c_3 \\ \Omega &= \Omega_0 + \delta\Omega + c_4 \\ i &= i_0 + \delta i + c_5 \\ \lambda_0 &= \lambda_0^0 + \delta\lambda_0 + c_6 \end{aligned}$$

where $a_0 + c_1$, $h_0 + c_2$, $l_0 + c_3$, $\Omega_0 + c_4$, $i_0 + c_5$ $\lambda_0^0 + c_6$ are six constants of integration These constants can be determined in various ways

First method We determine the constants of integration c_1, c_2, c_3, c_4, c_5, and c_6 in such a way that when $t = t_0$ (epoch of osculation) the perturbations $\delta a + c_1$, $\delta h + c_2$, $\delta l + c_3$, $\delta\Omega + c_4$, $\delta i + c_5$, and $\delta\lambda_0 + c_6$ become zero, that is, we assume

$$\begin{aligned} c_1 &= -(\delta a)_{t=t_0} & c_4 &= -(\delta\Omega)_{t=t_0} \\ c_2 &= -(\delta h)_{t=t_0} & c_5 &= -(\delta i)_{t=t_0} \\ c_3 &= -(\delta l)_{t=t_0} & c_6 &= -(\delta\lambda_0)_{t=t_0} \end{aligned}$$

Then

$$\begin{aligned} a &= a_0 + [\delta a - (\delta a)_{t=t_0}] \\ h &= h_0 + [\delta h - (\delta h)_{t=t_0}] \\ l &= l_0 + [\delta l - (\delta l)_{t=t_0}] \\ \Omega &= \Omega_0 + [\delta\Omega - (\delta\Omega)_{t=t_0}] \\ i &= i_0 + [\delta i - (\delta i)_{t=t_0}] \\ \lambda_0 &= \lambda_0^0 + [\delta\lambda_0 - (\delta\lambda_0)_{t=t_0}] \end{aligned} \quad (15)$$

Thus, the perturbations of the first-order in elements are calculated according to the formula

$$\begin{aligned} \delta_1 a &= \delta a - (\delta a)_{t=t_0} \\ \delta_1 h &= \delta h - (\delta h)_{t=t_0} \\ \delta_1 l &= \delta l - (\delta l)_{t=t_0} \\ \delta_1 \Omega &= \delta\Omega - (\delta\Omega)_{t=t_0} \\ \delta_1 i &= \delta i - (\delta i)_{t=t_0} \\ \delta_1 \lambda_0 &= \delta\lambda_0 - (\delta\lambda_0)_{t=t_0} \end{aligned} \quad (16)$$

and the same elements at any time t are obtained according to the formulas

$$\begin{aligned} a &= a_0 + \delta_1 a & \Omega &= \Omega_0 + \delta_1\Omega \\ h &= h_0 + \delta_1 h & i &= i_0 + \delta_1 i \\ l &= l_0 + \delta_1 l & \lambda_0 &= \lambda_0^0 + \delta_1\lambda_0 \end{aligned} \quad (17)$$

The orbital elements a, h, l, Ω, i, and λ_0 are called the osculating orbital elements The constants of integration $a_0, h_0, l_0, \Omega_0, i_0$, and λ_0^0 are called the osculating elements for the epoch of osculation ($t = t_0$) and are determined from observations

Second method The constants of integration c_1, c_2, c_3, c_4, c_5, and c_6 are assumed to be equal to zero Then

$$\begin{aligned} a &= a_0 + \delta a & \Omega &= \Omega_0 + \delta\Omega \\ h &= h_0 + \delta h & i &= i_0 + \delta i \\ l &= l_0 + \delta l & \lambda_0 &= \lambda_0^0 + \delta\lambda \end{aligned} \quad (18)$$

In this case a, h, l, Ω, i, and λ_0 are the osculating orbital elements at time t, and we will call the constants of integration $a_0, h_0, l_0, \Omega_0, i_0$, and λ_0^0 the mean orbital elements for the time t_0 The constants of integration $a_0, h_0, l_0, \Omega_0, i_0$, and λ_0^0 are determined from observations

Third method For elements having secular perturbations proportional to time we can write

$$\begin{aligned} h &= h_0 + At + \delta h + c_2 \\ \bar{l} &= l_0 + Bt + \delta l + c_3 \\ \Omega &= \Omega_0 + Ct + \delta\Omega + c_4 \\ \lambda_0 &= \lambda_0^0 + Dt + \delta\lambda_0 + c_6 \end{aligned} \quad (19)$$

whre A, B, C, and D are known constant coefficients, and δh,-δl, $\delta\Omega$, and $\delta\lambda_0$ contain only periodic terms We assume the constants of integration c_2, c_3, c_4, and c_6 to be equal to zero Then

$$\begin{aligned} h &= h_0 + At + \delta h \\ l &= l_0 + Bt + \delta l \\ \Omega &= \Omega_0 + Ct + \delta\Omega \\ \lambda_0 &= \lambda_0^0 + Dt + \delta\lambda_0 \end{aligned} \quad (20)$$

We adopt the notation

$$\begin{aligned} \bar{h} &= h_0 + At \\ \bar{l} &= l_0 + Bt \\ \bar{\Omega} &= \Omega_0 + Ct \\ \bar{\lambda}_0 &= \lambda_0^0 + Dt \end{aligned} \quad (21)$$

for the mean orbital elements for the time t Then

$$\begin{aligned} h &= \bar{h} + \delta h \\ l &= \bar{l} + \delta l \\ \Omega &= \bar{\Omega} + \delta\Omega \\ \lambda_0 &= \bar{\lambda}_0 + \delta\lambda_0 \end{aligned} \quad (22)$$

The elements h,l,Ω, and λ_0 are the osculating orbital elements The constants of integration h_0,l_0,Ω_0, and λ_0 are found from observations

It is obvious that there are other possible methods of determining the constants of integration The publication of the osculating orbital elements is convenient when the perturbations are small and the epoch of osculation is changed infrequently (small planets) For the orbits of artificial earth satellites this is not usually the case Furthermore, as we have seen, secular perturbations proportional to time appear among the orbital elements of satellites In the case of artificial satellites it is therefore advisable to determine the constants of integration by the third method and publish the mean orbital elements $a_0,\bar{h},\bar{l},i_0,\bar{\Omega}$, and $\bar{\lambda}_0$

5 Calculating the Satellite Coordinates

In calculating the coordinates of a satellite it is convenient to take directly, instead of λ_0, the longitude in the orbit for the time t:

$$\lambda = \lambda_0 + \int_{t_0}^{t} ndt = \lambda_0 + n_0(t - t_0) - \frac{3}{2}\frac{n_0}{a_0}\int_{t_0}^{t} \delta a dt \quad (23)$$

Introducing into (23) the expression for δa from (13) and integrating, we obtain

$$\begin{aligned} \lambda = {} & \lambda_0 + n_0(t - t_0) + \delta\lambda = \lambda_0 + n_0(t - t_0) + \\ & 2J\left(\frac{a'}{a}\right)^2\left(1 - \frac{3}{2}\sin^2 i\right)\left(nt + \frac{7}{4} l \sin\lambda - \frac{7}{4} h \cos\lambda\right) + \\ & 3J\left(\frac{a'}{a}\right)^2 \sin^2 i\left(-\frac{7}{24} h \cos\lambda - \frac{7}{24} l \sin\lambda + \frac{1}{4}\sin 2\lambda - \right. \\ & \left. \frac{49}{72} h \cos 3\lambda + \frac{49}{72} l \sin 3\lambda\right) + J\left(\frac{a'}{a}\right)^2 \cos^2 i \times \\ & \left(nt + \frac{7}{2} l \sin\lambda - \frac{5}{2} h \cos\lambda - \frac{1}{2}\sin 2\lambda - \right. \\ & \left. \frac{7}{6} l \sin 3\lambda + \frac{7}{6} h \cos 3\lambda\right) \end{aligned} \quad (24)$$

where on the right-hand sides we must insert the unperturbed elements $a_0,h_0,$, $\lambda = \lambda_0 + n_0(t - t_0)$ The osculating elements obtained can now be used to calculate the perturbed geocentric rectangular coordinates of the satellite For this purpose we adopt the well-known formulas:

$$\begin{aligned} x &= r \cos u \cos\Omega - r \sin u \sin\Omega \cos i \\ y &= r \cos u \sin\Omega + r \sin u \cos\Omega \cos i \\ z &= r \sin u \sin i \end{aligned} \quad (25)$$

where, to an accuracy of the order of the second power of h and l:

$$\begin{aligned} u &= \lambda + 2(l \sin\lambda - h \cos\lambda) + \frac{5}{4}(l^2 - h^2)\sin 2\lambda - \frac{5}{2} hl \cos 2\lambda \\ r &= a(1 - l \cos\lambda - h \sin\lambda) + a\left(\frac{h^2 + l^2}{2} - \frac{l^2 - h^2}{2}\cos 2\lambda - hl \sin 2\lambda\right) \end{aligned} \quad (26)$$

6 Improving the Orbital Elements

The topocentric satellite coordinates x', y', and z' are expressed by the following formulas:

$$\begin{aligned} x' &= x - X \\ y' &= y - Y \\ z' &= z - Z \end{aligned} \quad (27)$$

where X,Y, and Z are the geocentric coordinates of the observer:

$$\begin{aligned} X &= R \cos\varphi' \cos s \\ Y &= R \cos\varphi' \sin s \\ Z &= R \sin\varphi' \end{aligned} \quad (28)$$

Here φ' is the geocentric latitude of the observer, s is the local sidereal time for the observer, and R is the radius vector of the observer With an accuracy to and including the use of the first power of the eccentricity, we have

$$\begin{aligned} \Delta r &= (1 - l \cos\lambda - h \sin\lambda)\Delta a - a(\cos\lambda\Delta l + \sin\lambda\Delta h) + a(l \sin\lambda - h \cos\lambda)\Delta\lambda \\ \Delta u &= (1 + 2l \cos\lambda + 2h \sin\lambda)\Delta\lambda + 2 \sin\lambda\Delta l - 2\cos\lambda\Delta h \end{aligned} \quad (29)$$

Following the method published by Batrakov and Sochilina[1] we introduce a rotating system of coordinates in which the x axis is constantly directed to the ascending node of the satellite orbit Then formulas (27) take the form

$$\begin{aligned} x' &= r \cos u - X \\ y' &= r \sin u \cos i - Y \\ z' &= r \sin u \sin i - Z \end{aligned} \quad (30)$$

where

$$\begin{aligned} X &= R \cos\varphi' \cos(s - \Omega) \\ Y &= R \cos\varphi' \sin(s - \Omega) \\ Z &= R \sin\varphi' \end{aligned} \quad (31)$$

Then

$$\begin{aligned} \Delta x' &= \Delta r \cos u - r \sin u \,\Delta u - \Delta X \\ \Delta y' &= \Delta r \sin u \cos i + r \cos u \cos i \,\Delta u - r \sin u \sin i \,\Delta i - \Delta Y \\ \Delta z' &= \Delta r \sin u \sin i + r \cos u \sin i \,\Delta u + r \sin u \cos i \,\Delta i - \Delta Z \end{aligned} \quad (32)$$

where

$$\begin{aligned} \Delta X &= R \cos\varphi' \sin(s - \Omega)\Delta\Omega = Y\Delta\Omega \\ \Delta Y &= -R \cos\varphi' \cos(s - \Omega)\Delta\Omega = -X\Delta\Omega \\ \Delta Z &= 0 \end{aligned} \quad (33)$$

Substituting expressions (29) into (32), we obtain after some transformation

$$\begin{aligned} \Delta x' = {} & \frac{x}{r}(1 - l \cos\lambda - h \sin\lambda)\Delta a - \left(\frac{a}{r} x \cos\lambda + \frac{2}{\cos i} y \sin\lambda\right)\Delta l - \\ & \left(\frac{a}{r} x \sin\lambda - \frac{2}{\cos i} y \cos\lambda\right)\Delta h + \left[\frac{ax}{r}(l \sin\lambda - h \cos\lambda) - \frac{y}{\cos i}(1 + 2l \cos\lambda + 2h \sin\lambda)\right]\Delta\lambda - Y\Delta\Omega \end{aligned}$$

$$\Delta y' = \frac{y}{r}(1 - l \cos\lambda - h \sin\lambda)\Delta a -$$

$$\left(\frac{a}{r}\,y\cos\lambda - 2x\sin\lambda\cos i\right)\Delta l -$$
$$\left(\frac{a}{r}\,y\sin\lambda + 2x\cos\lambda\cos i\right)\Delta h +$$
$$\left[\frac{ay}{r}\,(l\sin\lambda - h\cos\lambda) + x\cos i(1 + 2l\cos\lambda + 2h\sin\lambda)\right]\times$$
$$\Delta\lambda - z\Delta i + X\Delta\Omega \quad (34)$$

$$\Delta z' = \frac{z}{r}\,(1 - l\cos\lambda - h\sin\lambda)\Delta a -$$
$$\left(\frac{a}{r}\,z\cos\lambda - 2x\sin\lambda\sin i)\right)\Delta l -$$
$$\left(\frac{a}{r}\,z\sin\lambda + 2x\cos\lambda\sin i\right)\Delta h + \left[\frac{az}{r}\,(l\sin\lambda - h\cos\lambda) +\right.$$
$$\left. x\sin i(1 + 2l\cos\lambda + 2h\sin\lambda)\right]\Delta\lambda + y\Delta i$$

where

$$x = r\cos u$$
$$y = r\sin u\cos i \quad (35)$$
$$z = r\sin u\sin i$$

are the geocentric satellite coordinates in a moving coordinate system The expressions for r and u are given by formulas (26) Using for brevity the notation

$$A = 1 - l\cos\lambda - h\sin\lambda$$
$$B = l\sin\lambda - h\cos\lambda \quad (36)$$
$$C = 1 + 2l\cos\lambda + 2h\sin\lambda$$

we obtain

$$\Delta x' = \frac{Ax}{r}\,\Delta a - \left(\frac{a}{r}\,x\cos\lambda + \frac{2}{\cos i}\,y\sin\lambda\right)\Delta l -$$
$$\left(\frac{a}{r}\,x\sin\lambda - \frac{2}{\cos i}\,y\cos\lambda\right)\Delta h + \left(\frac{Bax}{r} - \frac{Cy}{\cos i}\right)\Delta\lambda - Y\Delta\Omega$$
$$\Delta y' = \frac{Ay}{r}\,\Delta a - \left(\frac{a}{r}\,y\cos\lambda - 2x\sin\lambda\cos i\right)\Delta l -$$
$$\left(\frac{a}{r}\,y\sin\lambda + 2x\cos\lambda\cos i\right)\Delta h +$$
$$\left(\frac{Bay}{r} + Cx\cos i\right)\Delta\lambda - z\Delta i + X\Delta\Omega \quad (37)$$
$$\Delta z' = \frac{Az}{r}\,\Delta a - \left(\frac{a}{r}\,z\cos\lambda - 2x\sin\lambda\sin i\right)\Delta l -$$
$$\left(\frac{a}{r}\,z\sin\lambda + 2x\cos\lambda\sin i\right)\Delta h +$$
$$\left(\frac{Baz}{r} + Cx\sin i\right)\Delta\lambda + y\Delta i$$

In improving an orbit of small eccentricity we can assume in calculating the coefficients of the equations of condition that $e = 0$ Then $A = C = 1$, $B = 0$, and $r = a$ Formulas (37) take the simple form

$$\Delta x' = \frac{x}{a}\,\Delta a - \left(x\cos\lambda + \frac{2}{\cos i}\,y\sin\lambda\right)\Delta l -$$
$$\left(x\sin\lambda - \frac{2}{\cos i}\,y\cos\lambda\right)\Delta h - \frac{y}{\cos i}\,\Delta\lambda - Y\Delta\Omega$$
$$\Delta y' = \frac{y}{a}\,\Delta a - (y\cos\lambda - 2x\sin\lambda\cos i)\Delta l -$$
$$(y\sin\lambda + 2x\cos\lambda\cos i)\Delta h +$$
$$x\cos i\Delta\lambda - z\Delta i + X\Delta\Omega \quad (38)$$
$$\Delta z' = \frac{z}{a}\,\Delta a - (z\cos\lambda - 2x\sin\lambda\sin i)\Delta l -$$
$$(z\sin\lambda + 2x\cos\lambda\sin i)\Delta h + x\sin i\,\Delta\lambda + y\Delta i$$

We must now replace $\Delta\lambda$ by the correction $\Delta\lambda_0$ We have

$$\Delta\lambda = \Delta\lambda_0 + \Delta n\,t$$

or

$$\Delta\lambda = \Delta\lambda_0 - \frac{3}{2}\frac{n}{a}\,\Delta a\,t$$

Substituting into (38), we finally find

$$\Delta x' = \left(x - \frac{3}{2}\frac{ny}{\cos i}\,t\right)\frac{\Delta a}{a} - \left(x\cos\lambda + \frac{2}{\cos i}\,y\sin\lambda\right)\Delta l -$$
$$\left(x\sin\lambda - \frac{2}{\cos i}\,y\cos\lambda\right)\Delta h - \frac{y}{\cos i}\,\Delta\lambda_0 - Y\Delta\Omega$$
$$\Delta y' = \left(y - \frac{3}{2}nx\cos i\;t\right)\frac{\Delta a}{a} -$$
$$(y\cos\lambda - 2\cos i\;x\sin\lambda)\Delta l - (y\sin\lambda +$$
$$2\cos i\;x\cos\lambda)\Delta h + x\cos i\;\Delta\lambda_0 - z\Delta i + X\Delta\Omega \quad (39)$$
$$\Delta z' = \left(z - \frac{3}{2}nx\sin i\;t\right)\frac{\Delta a}{a} - (z\cos\lambda - 2\sin i\;x\sin\lambda)\Delta l -$$
$$(z\sin\lambda + 2\sin i\cdot x\cos\lambda)\Delta h + x\sin i\Delta\lambda_0 + y\Delta i$$

For $i \approx 90°$ in the formula for $\Delta x'$ we replace $y/\cos i$ by $z/\cos i$ Formulas (39) give us the corrections of the topocentric rectangular satellite coordinates by correcting the orbital elements $\Delta a, \Delta l, \Delta h, \Delta\Omega, \Delta i$, and $\Delta\lambda_0$ and the geocentric coordinates x, y, and z in a moving coordinate system (35) We now find the relation between the corrections of the equatorial satellite coordinate and those for the element of the orbit We have

$$\rho\cos(\alpha - \Omega)\cos\delta = x - X(\Omega) = x'$$
$$\rho\sin(\alpha - \Omega)\cos\delta = y - Y(\Omega) = y' \quad (40)$$
$$\rho\sin\delta = z - Z = z'$$

Hence after differentiating and performing some transformation we obtain

$$\rho\cos\delta\Delta(\alpha - \Omega) = -\sin(\alpha - \Omega)\Delta x' + \cos(\alpha - \Omega)\Delta y'$$
$$\rho\Delta\delta = -\sin\delta\cos(\alpha - \Omega)\Delta x' - \sin\delta\sin(\alpha - \Omega)\Delta y' + \cos\delta\Delta z'$$
$$\Delta\rho = \cos\delta\cos(\alpha - \Omega)\Delta x' + \cos\delta\sin(\alpha - \Omega)\Delta y' + \sin\delta\Delta z'$$

Since the correction $\Delta\rho$ cannot be determined directly from optical observations, we finally arrive at the following formulas which relate the corrections in the equatorial and rectangular topocentric coordinates:

$$\rho\cos\delta\;\Delta\alpha = -\sin(\alpha - \Omega)\Delta x' + \cos(\alpha - \Omega)\Delta y' + \rho\cos\delta\Delta\Omega \quad (41)$$
$$\rho\Delta\delta = -\sin\delta\cos(\alpha - \Omega)\Delta x' - \sin\delta\sin(\alpha - \Omega)\Delta y' + \cos\delta\;\Delta z'$$

where $\Delta x'$, $\Delta y'$, and $\Delta z'$ are related to corrections in the elements by formulas (39)

Thus, with the corrected elements a, h, l, Ω, i, and λ_0, we calculate the equatorial satellite coordinates in a rotating coordinate system To do so we use formulas (30), (31), and (35) A comparison of the calculated values of $(\alpha - \Omega)_c$ and δ with the observed values $(\alpha - \Omega)_0$ and δ_0 gives us the corrections for $\Delta\alpha$ and $\Delta\delta$ Then using formulas (39) and

(41), we find the corrections for the elements $\Delta a, \Delta h, \Delta l, \Delta\Omega, \Delta i$, and $\Delta\lambda_0$

The correction for the mean motion is easily found according to the formula

$$\Delta n = -\frac{3}{2} n \frac{\Delta a}{a} \tag{42}$$

The longitude in the orbit λ at time t is determined according to the formula

$$\lambda = (\lambda_0 + \Delta\lambda_0) + (n + \Delta n)t + \delta\lambda \tag{43}$$

where $\delta\lambda$ must be calculated according to formula (24)

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Estimate of Errors for Approximate Solutions of the Simplest Equations of Gasdynamics

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Introduction

In the present article we propose a new method for obtaining approximate generalized solutions for the simplest system of equations of gas dynamics:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial (p + \rho u^2)}{\partial x} = 0$$

with the equation of state

$$p = A\rho^\gamma \qquad A > 0 \qquad \gamma > 1$$

This method closely resembles the one used in (1)

It is shown that the constructed approximate solutions will approximately satisfy the differential equations in the weak sense, i e , for any sufficiently smooth finite function φ we have

$$\iint_{t>0} \left(\rho \frac{\partial \varphi}{\partial t} - \rho u \frac{\partial \varphi}{\partial x}\right) dxdt + \int_{t=0} \rho_0(x)\varphi(x,0)dx \to 0$$

$$\iint_{t>0} \left[\rho u \frac{\partial \varphi}{\partial t} - (p + \rho u^2)\frac{\partial \varphi}{\partial x}\right] dxdt + \int_{t=0} \rho_0(x)u_0(x)\varphi(x,0)dx \to 0$$

where $\rho_0(x)$, $u_0(x)$ are given functions (initial conditions) The foregoing sums of integrals tend to zero as a parameter h, characteristic of the accuracy of the system, decreases

We shall show that for positive $\varphi(x,t)$:

$$\iint_{t>0} \left[\rho\left(E + \frac{u^2}{2}\right)\frac{\partial \varphi}{\partial t} - \rho u\left(E + \frac{p}{\rho} + \frac{u^2}{2}\right)\frac{\partial \varphi}{\partial x}\right] dxdt + \int_{t=0} \rho_0\left[E(\rho_0) + \frac{u_0^2}{2}\right]\varphi(x,0)dx$$

will tend to some nonnegative constant as $h \to 0$ As a result, we can be sure that the limit of the approximate solutions (if it exists) cannot contain any inadmissible discontinuous rarefaction waves

We shall first construct the approximate solutions and show that they are bounded In this first part of our discussion we use weaker restrictions on the equation of state $p = p(\rho)$ In order to formulate these restrictions, we introduce the specific volume $V = 1/\rho$ instead of the density ρ and express the equation of state in the form $p = p(\rho) = p(1/V) = g(V)$

Let $g(V)$ satisfy the following conditions:

1) $g(V)$ is defined for all $V > 0$
2) $g'(V) < 0, g''(V) > 0$
3) $0 \leqslant g(V) \leqslant +\infty$; $g(V) \to +\infty$ as $V \to 0$ $g(V) \to 0$ as $V \to \infty$, $V\sqrt{-g'(V)} \to 0$ as $V \to \infty$
4) $\int^V g(V)dV$ is convergent as $V \to \infty$, so that we may set $E = E(\rho) = E(1/V) = -\int_\infty^V g(V)dV$
5) $\int^V \sqrt{-g'(V)}dV$ is convergent as $V \to \infty$ and divergent as $V \to 0$
6) $-(1/V)\int_\infty^V \sqrt{-g'(V)}dV$ is bounded if V is bounded below by a positive constant

All of these conditions are satisfied if the equation of state is of the form

$$p = A\rho^\gamma \qquad A > 0 \qquad \gamma > 1$$

In Sec 1 we state a number of inequalities resulting from our restrictions on the equation of state The first part of the paper will be based on these inequalities In Sec 7 the inequalities will be sharpened, using $p = A\rho^\gamma$ This refinement will enable us to estimate the variation of the solution, which appears in the error estimate

1 Inequalities

The inequalities which we shall state here involve, besides $p(\rho) = g(V), E(\rho)$, the following functions:

$$c(\rho) = V\sqrt{-g'(V)}$$

$$\Phi(\rho) = -\int_\infty^V \sqrt{-g'(V)}dV$$

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